



1.6.5 The absolute value of a complex number

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Introduction

- In this topic, we will
 - Define the absolute value of a complex number
 - Consider at a few properties
 - We will prove these properties (or theorems) rigorously





The absolute value

- For a real number x ,
 - $|x|$ is the absolute value or magnitude of that value
 - Some properties of the absolute value include:
 - $|x| \geq 0$
 - $|x| = 0$ if and only if $x = 0$
 - $|xy| = |x||y|$
- Is there an absolute value of a complex number?

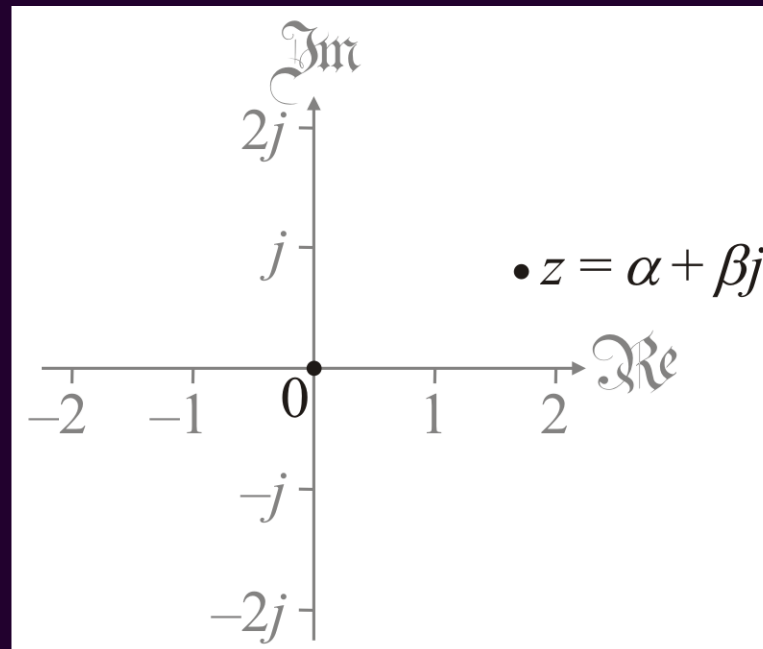




The absolute value

- Consider the geometric interpretation of the complex number

$$\alpha + \beta j$$



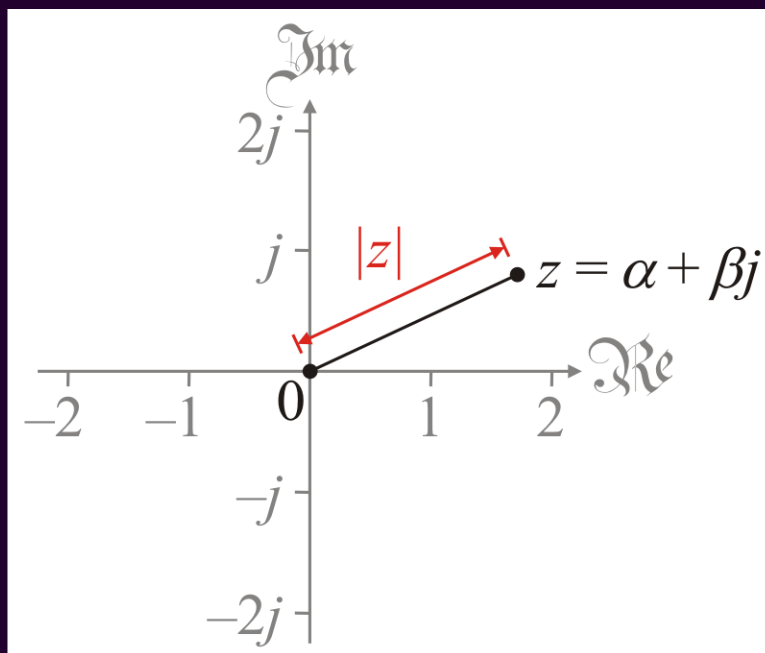


The absolute value

- Given this interpretation, we will use the distance to 0
 - For $z = \alpha + \beta j$,

$$|z| = \sqrt{\alpha^2 + \beta^2} = \sqrt{\Re(z)^2 + \Im(z)^2}$$

- Important: $\Im(z)$ is the real number β and not βj





The absolute value

- For example,
 - If $z_1 = 3 + 4j$, then
$$|z_1| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$$
 - If $z_2 = -2.4 - 0.7j$, then
$$|z_2| = \sqrt{(-2.4)^2 + (-0.7)^2} = \sqrt{5.76 + 0.49} = \sqrt{6.25} = 2.5$$
 - For any real θ , if $z_3 = \cos(\theta) + \sin(\theta)j$, then
$$|z_3| = \sqrt{\cos^2(\theta) + \sin^2(\theta)} = \sqrt{1} = 1$$
 - If $z_4 = 0.1931398164 - 0.02242417046j$, then
$$\begin{aligned} |z_4| &= \sqrt{0.1931398164^2 + (-0.02242417046)^2} \\ &\approx \sqrt{0.03780583210} \\ &\approx 0.1944372189 \end{aligned}$$





Absolute value of a real

Theorem: If z is real, then $|z| = |\Re(z)|$.

- Note, this says that the absolute value of a real complex number has the same value of the absolute value of the real part

Proof:

If z is real, then $\Im(z) = 0$, so $z = \alpha + 0j$ where $\Re(z) = \alpha$.

$$\begin{aligned}\text{Thus, } |z| &= \sqrt{\alpha^2 + 0^2} \\ &= \sqrt{\alpha^2} \\ &= |\alpha| = |\Re(z)|. \quad \text{QED}\end{aligned}$$

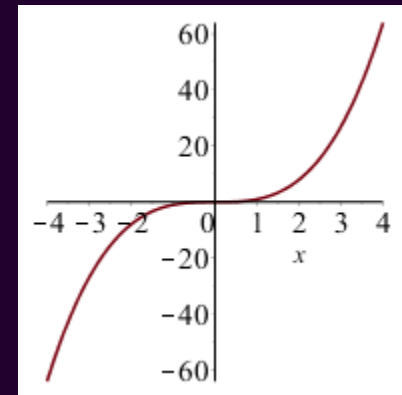




Absolute value is non-negative

Theorem: For any complex number z , $|z| \geq 0$.

- To prove this, we need a new concept:
 - A function f is *strictly monotonic increasing*
 $f(x) < f(y)$ whenever $x < y$
- Some examples:
 - $\sqrt{x}$ is strictly monotonic increasing for $x \geq 0$
 - $\ln(x)$ is strictly monotonic increasing for $x > 0$
 - e^x is strictly monotonic increasing for all real x
 - x^2 is strictly monotonic increasing for $x \geq 0$
 - x^3 is strictly monotonic increasing for all real x





Absolute value is non-negative

Theorem: For any complex number z , $|z| \geq 0$.

Proof:

If $z = \alpha + \beta j$, then α and β are real

Thus, $\alpha^2 \geq 0$ and $\beta^2 \geq 0$.

Thus, $\alpha^2 + \beta^2 \geq 0$.

But the square root is strictly monotonic increasing, so

$$|z| = \sqrt{\alpha^2 + \beta^2} \geq \sqrt{0} = 0. \quad \mathcal{QED}$$

$$\alpha^2 \geq 0$$

$$\beta^2 \geq 0$$

$$\hline \alpha^2 + \beta^2 \geq 0 + 0$$





Absolute value is positive definite

Theorem: $|z| = 0$ if and only if $z = 0$.

- How do we prove a statement that says P if and only if Q
 - There are four ways of proving this:
 1. Assume P is true, and show that Q is true.
Assume Q is true, and show that P is true.
 2. Assume P is true, and show that Q is true.
Assume P is false, and show that Q is false.
 3. Assume P is false, and show that Q is false.
Assume Q is false, and show that P is false.
 4. Assume Q is true, and show that P is true.
Assume Q is false, and show that P is false.





Absolute value is positive definite

Theorem: $|z| = 0$ if and only if $z = 0$.

Proof: Assume $z = 0 + 0j$.

$$\text{Thus, } |z| = \sqrt{0^2 + 0^2} = \sqrt{0} = 0.$$

Assume $z \neq 0 + 0j$.

Thus, $z = \alpha + \beta j$ where $\alpha \neq 0$ or $\beta \neq 0$.

Thus, either $\alpha^2 > 0$ and $\beta^2 \geq 0$, or $\alpha^2 \geq 0$ and $\beta^2 > 0$.

– If $\beta^2 \geq 0$, $\alpha^2 + \beta^2 \geq \alpha^2$, yet $\alpha^2 > 0$

– If $\alpha^2 \geq 0$, $\alpha^2 + \beta^2 \geq \beta^2$, yet $\beta^2 > 0$

Thus, in either case, $\alpha^2 + \beta^2 > 0$.

But the square root is strictly monotonic increasing

$$\text{Thus, } |z| = \sqrt{\alpha^2 + \beta^2} > \sqrt{0} = 0. \quad \mathcal{QED}$$





Absolute value is absolutely scalable

- To this point, if γ is real and $z = \alpha + \beta j$, we have defined γz

- Recall that,

- $z = \alpha + \beta j = \alpha + \beta\sqrt{-1}$

- For a real number γ , we have:

$$\gamma z = \gamma(\alpha + \beta j) = \gamma(\alpha + \beta\sqrt{-1}) = \gamma\alpha + \gamma\beta\sqrt{-1} = \gamma\alpha + \gamma\beta j$$

- Question: Is $|\gamma z| = |\gamma||z|$?





Absolute value is absolutely scalable

Theorem: $|\gamma z| = |\gamma| |z|$

Proof: $|\gamma z| = |\gamma\alpha + \gamma\beta j|$

$$= \sqrt{(\gamma\alpha)^2 + (\gamma\beta)^2}$$

$$= \sqrt{\gamma^2 \alpha^2 + \gamma^2 \beta^2}$$

$$= \sqrt{\gamma^2 (\alpha^2 + \beta^2)}$$

but $\gamma^2 \geq 0$

$$= \sqrt{\gamma^2} \sqrt{\alpha^2 + \beta^2}$$

$$= |\gamma| |z| \quad \text{QED}$$





Absolute value is absolutely scalable

- A few examples:
 - If $z = 3 + 4j$ then $|z| = 5$
 - Then $-3z = -9 - 12j$
 - Thus, $|-3||z| = 3 \times 5 = 15$
 - Also, $|-9 - 12j| = \sqrt{(-9)^2 + (-12)^2} = \sqrt{81 + 144} = \sqrt{225} = 15$
 - If $w = 0.8 + 1.5j$, then $|w| = 1.7$
 - Then, $2.5w = 2 + 3.75j$
 - Thus, $|2.5||w| = 2.5 \times 1.7 = 4.25$
 - Also, $|2 + 3.75j| = \sqrt{2^2 + 3.75^2} = \sqrt{4 + 14.0625} = \sqrt{18.0625} = 4.25$





Summary

- In this topic, we've introduced the absolute value of a complex number
 - The distance to 0
 - We defined functions that are strictly monotonic increasing
 - We saw that:
 - $|z| \geq 0$
 - $|z| = 0$ if and only if $z = 0$
 - $|\gamma z| = |\gamma||z|$ if γ is real
 - Reminder: we will be seeing other proofs in this course
 - This is just a taste of how we rigorously prove these theorems





References

- [1] https://en.wikipedia.org/wiki/Absolute_value#Complex_numbers





Acknowledgments

None so far.





Colophon

These slides were prepared using the Cambria typeface. Mathematical equations use Times New Roman, and source code is presented using Consolas.

The photographs of flowers and a monarch butter appearing on the title slide and accenting the top of each other slide were taken at the Royal Botanical Gardens in October of 2017 by Douglas Wilhelm Harder. Please see

<https://www.rbg.ca/>

for more information.





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